Quantum Computing

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Review: Lecture 4

- Quantum States
  - Complex/probability/transition amplitudes
- Observables & Measuring
  - The observables are represented by Hermitian matrices
  - The possible results of a measurement are the eigenvalues of the observable matrices. If the system is in the eigenstate, the measurement result is guaranteed to be the related eigenvalue
  - Unambiguously distinguishable states are represented by orthogonal vectors
  - The observing probability is the modulus square of the probability amplitude
- Dynamics
  - The evolution of a quantum system is given by a unitary matrix

Lecture 5: Composite Systems

1

Tensor Product of Vector Space

- Definition
- Examples
- Properties

2

Assembling Systems

- Assembling classical probabilistic system
- Tensor product of state vectors and operator matrices
- keynotes

3

Assembling Quantum System

- The principle (cont.)
- Entanglement and entangled states
- Entangled composite spin system
- keynotes

4

世纪之争

- 玻爱之争
- EPR佯谬
- Bell不等式
- CHSH不等式

1. Tensor Product

- Definition: tensor product
  - $\mathbb{V}$ has a basis $\mathfrak{B} = \{\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n-1}\}$
    - A vector $\boldsymbol{v} \in \mathbb{V}$ can be represented as $\boldsymbol{v} = \sum_{i=0}^{n-1} c_i \boldsymbol{e}_i$
  - $\mathbb{V}'$ has a basis $\mathfrak{B}' = \{\boldsymbol{e}'_0, \boldsymbol{e}'_1, \cdots, \boldsymbol{e}'_{m-1}\}$
    - A vector $\boldsymbol{v}' \in \mathbb{V}'$ can be represented as $\boldsymbol{v}' = \sum_{j=0}^{m-1} c'_j \boldsymbol{e}'_j$
  - $\boldsymbol{v} \otimes \boldsymbol{v}'$ is defined as

$$\boldsymbol{v} \otimes \boldsymbol{v}' = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (c_i \times c'_j)\,(\boldsymbol{e}_i \otimes \boldsymbol{e}'_j)$$

1. Tensor Product

- Equation: tensor product for matrices

$$\otimes:\ \mathbb{C}^{m\times m'}\times\mathbb{C}^{n\times n'}\rightarrow\mathbb{C}^{mn\times m'n'}$$

$$(\mathbf{A}\otimes\mathbf{B})\,(j,\ k)=\mathbf{A}(j/n,\ k/n')\times\mathbf{B}(j\ \text{ mod n},\ \text{k mod n}')$$

1. Tensor Product

- Example: tensor product of matrices

We will need to know not only how to take the tensor product of two vectors, but also how to determine the **tensor product of two matrices.**[10] Consider two matrices

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix}. \tag{2.172}$$

From the association given in Equation (2.165), it can be seen that the tensor product $A \otimes B$ is the matrix that has every element of A, scalar multiplied with the entire matrix B. That is,

$$A \otimes B = \begin{bmatrix} a_{0,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{0,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{1,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{0,0} \times b_{0,0} & a_{0,0} \times b_{0,1} & a_{0,0} \times b_{0,2} & a_{0,1} \times b_{0,0} & a_{0,1} \times b_{0,1} & a_{0,1} \times b_{0,2} \\ a_{0,0} \times b_{1,0} & a_{0,0} \times b_{1,1} & a_{0,0} \times b_{1,2} & a_{0,1} \times b_{1,0} & a_{0,1} \times b_{1,1} & a_{0,1} \times b_{1,2} \\ a_{0,0} \times b_{2,0} & a_{0,0} \times b_{2,1} & a_{0,0} \times b_{2,2} & a_{0,1} \times b_{2,0} & a_{0,1} \times b_{2,1} & a_{0,1} \times b_{2,2} \\ a_{1,0} \times b_{0,0} & a_{1,0} \times b_{0,1} & a_{1,0} \times b_{0,2} & a_{1,1} \times b_{0,0} & a_{1,1} \times b_{0,1} & a_{1,1} \times b_{0,2} \\ a_{1,0} \times b_{1,0} & a_{1,0} \times b_{1,1} & a_{1,0} \times b_{1,2} & a_{1,1} \times b_{1,0} & a_{1,1} \times b_{1,1} & a_{1,1} \times b_{1,2} \\ a_{1,0} \times b_{2,0} & a_{1,0} \times b_{2,1} & a_{1,0} \times b_{2,2} & a_{1,1} \times b_{2,0} & a_{1,1} \times b_{2,1} & a_{1,1} \times b_{2,2} \end{bmatrix}. \tag{2.173}$$

1. Tensor Product

- Example: tensor product of vectors

from the association given in Equation (2.165) that the **tensor product of vectors** is defined as follows:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_1 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_2 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_3 \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_0 b_2 \\ a_1 b_0 \\ a_1 b_1 \\ a_1 b_2 \\ a_2 b_0 \\ a_2 b_1 \\ a_2 b_2 \\ a_3 b_0 \\ a_3 b_1 \\ a_3 b_2 \end{bmatrix}. \tag{2.166}$$

1. Tensor Product

- Properties
  - Associativity :

$$\mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C}$$

  - Tensor product respects the adjoint :

$$(\mathbf{A} \otimes \mathbf{B})^{\dagger} = \mathbf{A}^{\dagger} \otimes \mathbf{B}^{\dagger}$$

  - Tensor product allows “parallel action” :

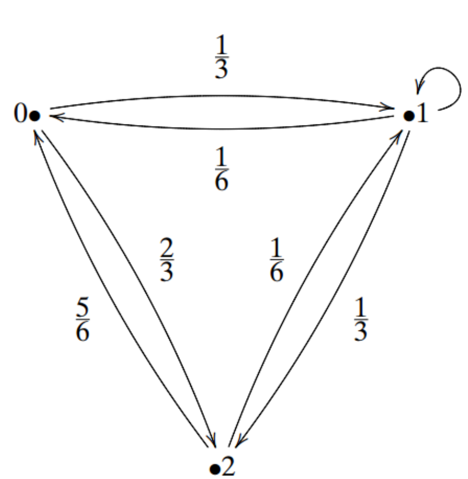
$$(\mathbf{A} \times \boldsymbol{v}) \otimes (\mathbf{B} \times \boldsymbol{w}) = (\mathbf{A} \otimes \mathbf{B}) \times (\boldsymbol{v} \otimes \boldsymbol{w})$$

2. Assembling Systems

- Assembling classical probabilistic systems
  - Graphs and matrices

Red marble $\in$ {0, 1, 2}

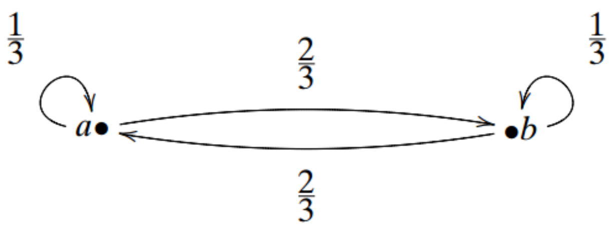
$G_M =$



$$M = \begin{bmatrix} 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

Blue marble $\in$ {a, b}

$G_N =$



$$N = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

2. Assembling Systems

- State
  - 3 states for G_M system: 0, 1 and 2
  - 2 states for G_N system: a and b
  - 3×2=6 states in the combined system
- Example

Is this a tensor product of a 3×1 vector with a 2×1 vector ?

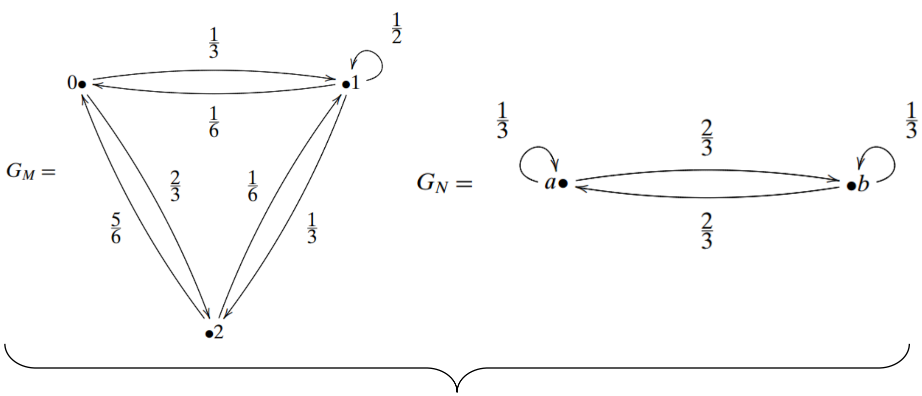
$$X = \begin{matrix} 0a \\ 0b \\ 1a \\ 1b \\ 2a \\ 2b \end{matrix} \begin{bmatrix} \frac{1}{18} \\ 0 \\ \frac{2}{18} \\ \frac{1}{3} \\ 0 \\ \frac{1}{2} \end{bmatrix} \Rightarrow$$

$\frac{1}{18}$ chance of the red marble being on vertex 0 and the blue marble being on vertex a,
0 chance of the red marble being on vertex 0 and the blue marble being on vertex b,
$\frac{2}{18}$ chance of the red marble being on vertex 1 and the blue marble being on vertex a,
$\frac{1}{3}$ chance of the red marble being on vertex 1 and the blue marble being on vertex b,
0 chance of the red marble being on vertex 2 and the blue marble being on vertex a, and
$\frac{1}{2}$ chance of the red marble being on vertex 2 and the blue marble being on vertex b.

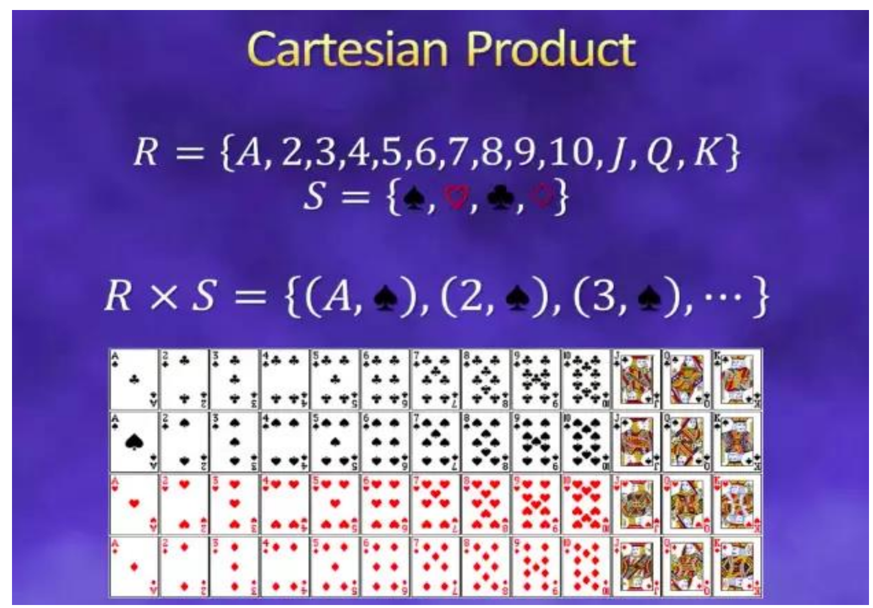
2. Assembling Systems

- Dynamics
  - Graph
    - Cartesian product

operation for set



$$G_{\mathbf{M}} \times G_{\mathbf{N}} = \{(0,\boldsymbol{a}),\ (0,\boldsymbol{b}),(1,\boldsymbol{a}),(1,\boldsymbol{b}),(2,\boldsymbol{a}),(2,\boldsymbol{b})\}$$



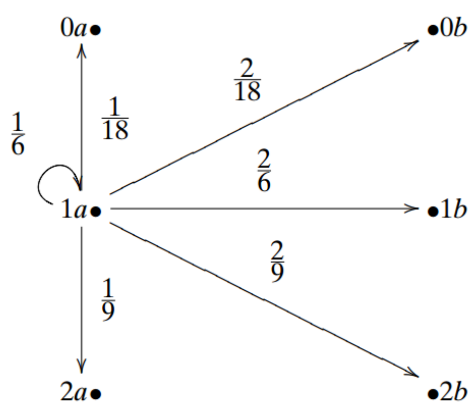
Source: 《学习笔记< Cartesian product >》 https://www.jianshu.com/p/3c866bee7b5e

2. Assembling Systems

- Dynamics
  - Matrix
    - tensor product $\mathbf{M}\otimes\mathbf{N}$: $\quad ij \xrightarrow{M[i',i]\times N[j',j]} i'j'$
  - Example

$$M\otimes N=\begin{array}{c}\\ \mathbf{0}\\ \mathbf{1}\\ \mathbf{2}\end{array}\begin{array}{c}\begin{array}{ccc}\mathbf{0} & \mathbf{1} & \mathbf{2}\end{array}\\ \left[\begin{array}{ccc}0\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & \frac{1}{6}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & \frac{5}{6}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}\\ \frac{1}{3}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & \frac{1}{2}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & \frac{1}{6}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}\\ \frac{2}{3}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & \frac{1}{3}\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix} & 0\begin{bmatrix}\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}\end{array}\right]\end{array} = \begin{array}{c}\mathbf{0a}\\\mathbf{0b}\\\mathbf{1a}\\\mathbf{1b}\\\mathbf{2a}\\\mathbf{2b}\end{array}\begin{array}{c}\begin{array}{cccccc}\mathbf{0a}&\mathbf{0b}&\mathbf{1a}&\mathbf{1b}&\mathbf{2a}&\mathbf{2b}\end{array}\\ \begin{bmatrix}0&0&\frac{1}{18}&\frac{2}{18}&\frac{5}{18}&\frac{10}{18}\\ 0&0&\frac{2}{18}&\frac{1}{18}&\frac{10}{18}&\frac{5}{18}\\ \frac{1}{9}&\frac{2}{9}&\frac{1}{6}&\frac{2}{6}&\frac{1}{18}&\frac{2}{18}\\ \frac{2}{9}&\frac{1}{9}&\frac{2}{6}&\frac{1}{6}&\frac{2}{18}&\frac{1}{18}\\ \frac{2}{9}&\frac{4}{9}&\frac{1}{9}&\frac{2}{9}&0&0\\ \frac{4}{9}&\frac{2}{9}&\frac{2}{9}&\frac{1}{9}&0&0\end{bmatrix}\end{array}$$

2. Assembling Systems

- Dynamics
  - Graph vs. Matrix
    - Cartesian product $G_{\mathbf{M}} \times G_{\mathbf{N}}$
    - Tensor product $\mathbf{M} \otimes \mathbf{N}$
  - Example
    - 

	0a	**0b**	**1a**	**1b**	**2a**	**2b**
0a	0	0	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{5}{18}$	$\frac{10}{18}$
0b	0	0	$\frac{2}{18}$	$\frac{1}{18}$	$\frac{10}{18}$	$\frac{5}{18}$
1a	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{18}$	$\frac{2}{18}$
1b	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{18}$	$\frac{1}{18}$
2a	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	0
2b	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	0	0

2. Assembling Systems

- Remarks
  - States
    - Tensor product from subsystems' state vectors
    - Entangled states (more interesting!)
  - Dynamics/matrices
    - Tensor product from subsystems' dynamics matrices
    - Other actions

2. Assembling Systems

- Higher order assembling systems
  - Assemble m n-vertex graph G
    - Graph: with n^m vertices

$$G^m = \underbrace{G \times G \times \cdots \times G}_{m \text{ times}}$$

    - Matrix: size of n^m -by- n^m

$$M_G^{\otimes m} = \underbrace{M_G \otimes M_G \otimes \cdots \otimes M_G}_{m \text{ times}}$$

2. Assembling Systems

- Keynotes
  - A composite system is represented by the Cartesian product of the transition graphs of its subsystems
  - If two matrices act on the subsystems independently, then their tensor product acts on the states of their combined system, i.e., $(\mathbf{M} * \boldsymbol{v}) \otimes (\mathbf{N} * \boldsymbol{w}) = (\mathbf{M} \otimes \mathbf{N}) * (\boldsymbol{v} \otimes \boldsymbol{w})$
  - There is an exponential growth in the amount of resources needed to describe larger and larger composite systems

3. Assembling Quantum System

- Principle
  - Assume we have two independent quantum systems $\mathcal{Q}$ and $\mathcal{Q}'$, represented respectively by the vector spaces $\mathbb{V}$ and $\mathbb{V}'$. The quantum system obtained by merging $\mathcal{Q}$ and $\mathcal{Q}'$ will have the tensor product $\mathbb{V} \otimes \mathbb{V}'$ as a state space.

3. Assembling Quantum System

- Example

The tensor product of vector spaces is associative, so we can progressively build larger and larger systems:

$$\mathbb{V}_0 \otimes \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_k. \tag{4.98}$$

Let us go back to our example. To begin with, there are $n \times m$ possible basic states:

$|x_0\rangle \otimes |y_0\rangle$, meaning the first particle is at x_0 and the second particle at y_0.
$|x_0\rangle \otimes |y_1\rangle$, meaning the first particle is at x_0 and second particle at y_1.
$\vdots$
$|x_0\rangle \otimes |y_{m-1}\rangle$, meaning the first particle is at x_0 and the second particle at y_{m-1}.
$|x_1\rangle \otimes |y_0\rangle$, meaning the first particle is at x_1 and the second particle at y_0.
$\vdots$
$|x_i\rangle \otimes |y_j\rangle$, meaning the first particle is at x_i and the second particle at y_j.
$\vdots$
$|x_{n-1}\rangle \otimes |y_{m-1}\rangle$, meaning the first particle is at x_{n-1} and the second particle at y_{m-1}.

Now, let us write the generic state vector as a superposition of the basic states:

$$|\psi\rangle = c_{0,0}|x_0\rangle \otimes |y_0\rangle + \cdots + c_{i,j}|x_i\rangle \otimes |y_j\rangle + \cdots + c_{n-1,m-1}|x_{n-1}\rangle \otimes |y_{m-1}\rangle, \tag{4.99}$$

which is a vector in the $(n \times m)$-dimensional complex space $\mathbb{C}^{n\times m}$.

3. Assembling Quantum System

- Entanglement
  - The condition that a state vector of an assembling system could not be written as the tensor product of basic states of its constituents.

3. Assembling Quantum System

- Example： two-particle system
  - Each particle is allow only two points

$$|x\rangle = c_0|x_0\rangle + c_1|x_1\rangle \qquad |y\rangle = c_0|y_0\rangle + c_1|y_1\rangle$$

  - The following state cannot be written as a tensor product

$$|\psi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$$

3. Assembling Quantum System

- Example： two-particle system
  - Why?

$$|x\rangle \otimes |y\rangle = (c_0|x_0\rangle + c_1|x_1\rangle) \otimes (c_0'|y_0\rangle + c_1'|y_1\rangle)$$

$$= \boxed{c_0 c_0'}|x_0\rangle \otimes |y_0\rangle + \boxed{c_0 c_1'}|x_0\rangle \otimes |y_1\rangle$$

$$+ \boxed{c_1 c_0'}|x_1\rangle \otimes |y_0\rangle + \boxed{c_1 c_1'}|x_1\rangle \otimes |y_1\rangle$$

$$= |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$$

$$\Rightarrow c_0 c_1' = c_1 c_0' = 0 \;\; \text{and} \;\; c_0 c_0' = c_1 c_1' = 1$$

3. Assembling Quantum System

- Explanation

$$|\psi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$$

has a 50–50 chance of being found at the position x_0 or at x_1. So, what if it is, in fact, found in position x_0? Because the term $|x_0\rangle \otimes |y_1\rangle$ has a 0 coefficient, we know that there is no chance that the second particle will be found in position y_1. We must then conclude that the second particle can only be found in position y_0. Similarly, if the first particle is found in position x_1, then the second particle must be in position y_1. Notice that the situation is perfectly symmetrical with respect to the two particles, i.e., it would be the same if we measured the second one first. The individual states of the two particles are intimately related to one another, or **entangled**. The amazing side of this story is that the x_i's can be light years away from the y_i's. Regardless of their actual distance in space, a measurement's outcome for one particle will always determine the measurement's outcome for the other one.

3. Assembling Quantum System

- Separable states
  - States that can be broken into the tensor product of states from the constituent subsystems are called Separable states

$$|\psi'\rangle = 1|x_0\rangle \otimes |y_0\rangle + 1|x_0\rangle \otimes |y_1\rangle + 1|x_1\rangle \otimes |y_0\rangle + 1|x_1\rangle \otimes |y_1\rangle$$

- Entangled states
  - states that are unbreakable are referred to as entangled states $|\psi\rangle = |x_0\rangle \otimes |y_0\rangle + |x_1\rangle \otimes |y_1\rangle$

3. Assembling Quantum System

- Entangled composite spin system
  - Basic states

$$\{\uparrow_L \otimes \uparrow_R, \quad \uparrow_L \otimes \downarrow_R, \quad \downarrow_L \otimes \uparrow_R, \quad \downarrow_L \otimes \downarrow_R\}$$

  - Entangled states

$$\frac{|\uparrow_L \otimes \downarrow_R\rangle + |\downarrow_L \otimes \uparrow_R\rangle}{\sqrt{2}}$$

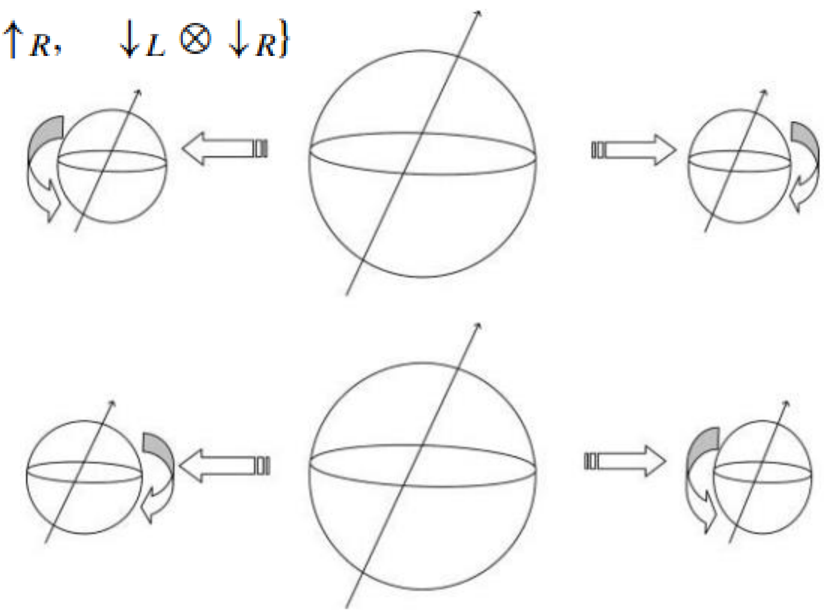


Figure 4.8. Two possible scenarios of a composite system where the total spin is zero.

3. Assembling Quantum System

- Keynotes
  - We can use the tensor product to build complex quantum systems out of simpler ones.
  - The new entangled system cannot be analyzed simply in terms of states belonging to its subsystems. An entire set of new entangled states has been created, which cannot be resolved into their constituents.

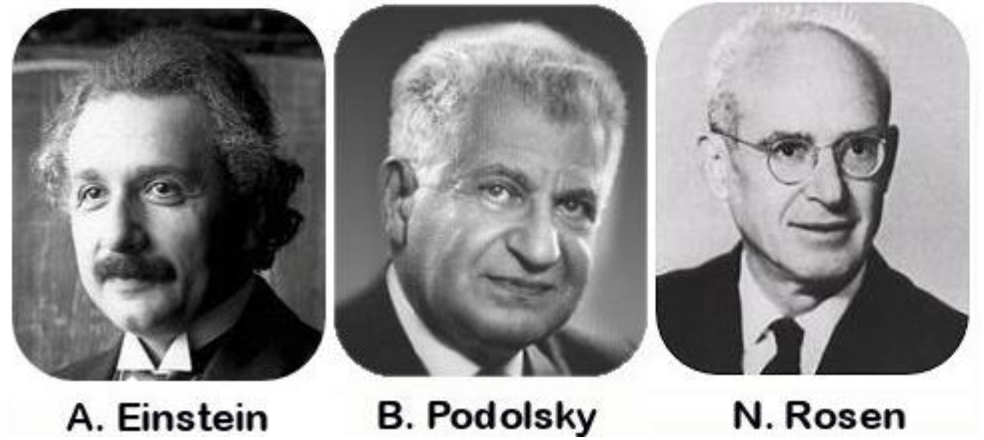
Athena Exclamation

4. 世纪之争

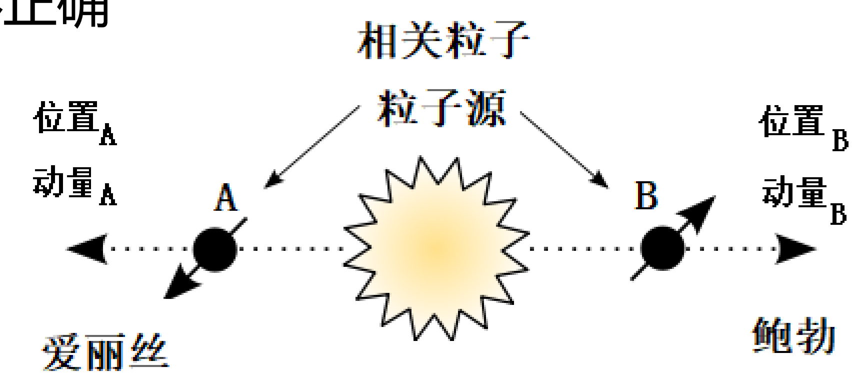
- 玻爱之争
  - 第五、六次索尔维会议（1927，1930）
  - EPR佯谬论文（1935）
- 爱因斯坦的质疑
  - 确定性（上帝不掷骰子）vs. 海森堡不确定原理
  - 实在性（物质世界的存在不依赖于观测手段）
  - 局域性（不可能有瞬时的超距作用）

参考资料：张天蓉，世纪幽灵：走进量子纠缠（第二版），中国科学技术大学出版社，2020年

4. 世纪之争



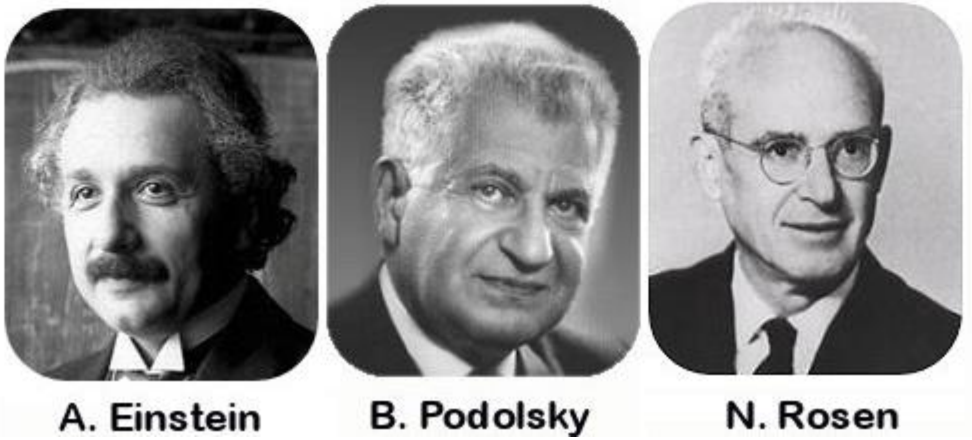
- EPR佯谬
  - 版本1（上帝不掷骰子）
    - 假设：A和B相互纠缠，二者的位置和动量都保持等值反号
    - 步骤 1：先测A的动量，则知B的动量
    - 步骤 2：再测B的位置
    - 步骤 3：根据步骤1和2，同时知道B的准确位置和动量
    - 结论：海森堡不确定原理不正确



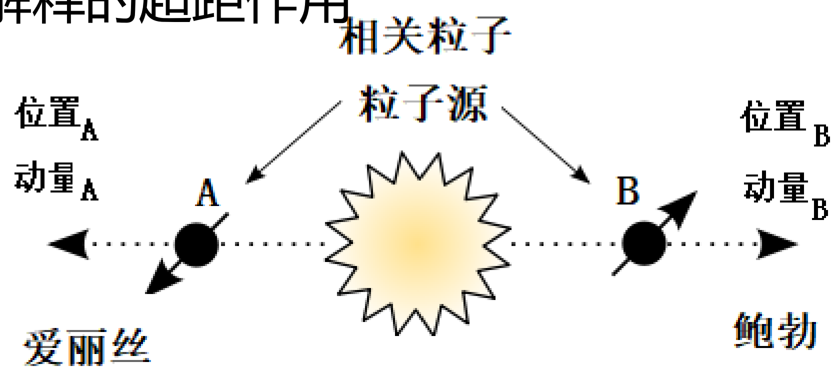
参考资料：

1. 埋人专业户，对EPR佯谬的理解与原文的忠实翻译，量子客，https://www.qtumist.com/post/8389
2. 张天蓉，量子理论的诞生和发展(16): EPR佯谬和玻尔的反击，http://www.360doc.com/content/20/0521/22/35201910_913772509.shtml

4. 世纪之争



- EPR佯谬
  - 版本2（诡异的超距作用，隐变量）
    - 假设：A和B相互纠缠，二者的位置和动量都保持等值反号
    - 步骤 1：观测A的自旋为上（下）
    - 步骤 2：因为守恒，B的自旋为下（上）
    - 问题：A和B相隔遥远，如何能及时通信?
    - 解释 1：存在现有物理无法解释的超距作用
    - 解释 2：隐变量



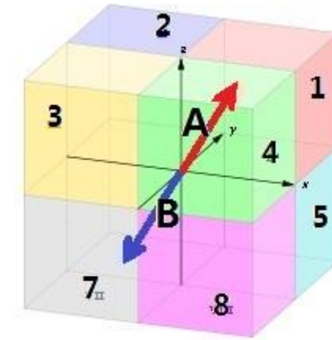
参考资料：

1. 张天蓉，世纪幽灵：走进量子纠缠（第二版），中国科学技术大学出版社，2020年
2. Umesh Vazirani, Lecture 1: Axioms of QM + Bell Inequalities, Lecture note, CS294

4. 世纪之争

- 隐变量解释
  - 量子纠缠虽然看起来是随机的，但却可能实在两粒子分离的那一刻就（由某种隐变量）决定好了的（如同把鞋子的左右脚放到两个不同的黑盒子中，然后被带到南北两极，再打开盒子一样）
  - 即使不知道隐变量是什么，也可以假设粒子的可观测量是这些隐变量的某种函数的统计平均值

4. 世纪之争



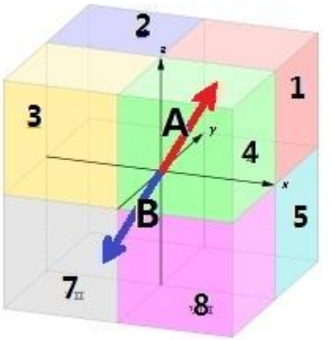
八个卦限中纠缠态粒子A,B的自旋

- Bell不等式
  - 由于A、B的纠缠性，红矢和蓝矢指向相反
  - 假设红矢出现在八个卦限中的概率分别为$n_1, \ldots, n_8$，有$n_1+n_2+n_3+n_4+n_5+n_6+n_7+n_8 = 1$
  - A、B纠缠，互为关联，定义关联函数
    - Pxx(L)：观察x方向红矢的符号，和x方向蓝矢的符号，如果两个符号相同，Pxx(L)的值就为+1；否则，Pxx(L)的值就为-1

参考资料：张天蓉，走近量子纠缠-7-贝尔不等式，https://blog.sciencenet.cn/blog-677221-537543.html

4. 世纪之争

- Bell不等式



八个卦限中纠缠态粒子A,B的自旋

L	Ax Ay Az（红矢）			Bx By Bz（蓝矢）			P	Pxx (L)	Pxz (L)	Pzy (L)	Pxy (L)
1	+	+	+	−	−	−	n1	-1	-1	-1	-1
2	−	+	+	+	−	−	n2	-1	+1	-1	+1
3	−	−	+	+	+	−	n3	-1	+1	+1	-1
4	+	−	+	−	+	−	n4	-1	-1	+1	+1
5	+	+	−	−	−	+	n5	-1	+1	+1	-1
6	−	+	−	+	−	+	n6	-1	-1	+1	+1
7	−	−	−	+	+	+	n7	-1	-1	-1	-1
8	+	−	−	−	+	+	n8	-1	+1	-1	+1

AB纠缠态自旋矢量八种可能性　　四个相关函数的值

Pxx代表的是A和B都从x方向观测时，它们的符号的平均相关性。因为纠缠的原因，A、B的符号总是相反的，所以同被在x方向观察时，它们的平均相关性是-1，即反相关

$$Pxx = -n1-n2-n3-n4-n5-n6-n7-n8 = -1$$

$$Pxz = -n1+n2+n3-n4+n5-n6-n7+n8$$

$$Pzy = -n1-n2+n3+n4+n5+n6-n7-n8$$

$$Pxy = -n1+n2-n3+n4-n5+n6-n7+n8$$

关联函数的平均值

Pxy代表的是从x方向观测A，从y方向观测B时，它们符号的平均相关性。如果自旋在每个方向的概率都一样，即：n1=n2=...n8=1/8的话，我们会得到Pxy为0。对Pzy和Pxz，也得到相同的结论。

当概率均等时，如在相同方向测量A、B的自旋，应该反相关；

而如果在不同方向测量A和B的自旋，平均来说应该不相关。

参考资料：张天蓉，走近量子纠缠-7-贝尔不等式，https://blog.sciencenet.cn/blog-677221-537543.html

4. 世纪之争

```
Pxx = -n1-n2-n3-n4-n5-n6-n7-n8 = -1
Pxz = -n1+n2+n3-n4+n5-n6-n7+n8
Pzy = -n1-n2+n3+n4+n5+n6-n7-n8
Pxy = -n1+n2-n3+n4-n5+n6-n7+n8
```

关联函数的平均值

- Bell不等式

$$
\begin{aligned}
|P_{xz} - P_{zy}| &= 2|n_2 - n_4 - n_6 + n_8| \\
&= 2|(n_2 + n_8) - (n_4 + n_6)| \\
&\leqslant 2(n_2 + n_4 + n_6 + n_8) \\
&= (n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8) \\
&\quad + (-n_1 + n_2 - n_3 + n_4 - n_5 + n_6 - n_7 + n_8) \\
&= 1 + P_{xy}
\end{aligned}
$$

在经典的框架下，这三个关联函数之间要满足的约束条件

参考资料：张天蓉，走近量子纠缠-7-贝尔不等式，https://blog.sciencenet.cn/blog-677221-537543.html

贝尔不等式：$|P_{xz}-P_{zy}|\leqslant 1+P_{xy}$

4. 世纪之争

- Bell不等式的缺点
  - 三个测量方向x, y, z是测量两个互为纠缠的粒子所共用的，而同时我们希望两个纠缠粒子分开得越远越好。上述两点很难同时满足
    - 间隔太远了，观测两个粒子所采用的xyz坐标系难对齐
  - Bell不等式假设两个纠缠粒子需要准确的反向飞行，这在真实实验中不可能完全满足

参考资料：张天蓉，世纪幽灵：走进量子纠缠（第二版），中国科学技术大学出版社，2020年

4. 世纪之争

- CHSH不等式
  - 在经典中，假设量子纠缠是不存在，一个系统中的量子坍缩的结果是独立的
  - 假设现在有两个粒子，一个给Alice，一个给Bob，Alice测量方法用Q或者R，Bob测量方法用S或者T，用哪种方法随机（可以通过扔骰子的方式），并且得到的结果一定是-1和+1中的一个

参考资料：yangdaixian，CHSH不等式以及它对量子力学的贡献，https://blog.csdn.net/yangdaixian/article/details/110287686

4. 世纪之争

- CHSH不等式
  - 两个纠缠例子
    - 分给Alice和Bob
  - Alice测量方法
    - Q or R
  - Bob测量方法
    - S or T
  - 用那种方法随机
  - 测量结果-1 or +1

我们现在看下面的式子 $QS + RS + RT - QT = (Q+R)S + (R-Q)T$ ，因为R和Q都只可能是+1和-1的一种，所以 $(Q+R)S = 0 \ or \ (R-Q)T = 0$ ，同时，我们也很容易能看出， $(Q+R)S + (R-Q)T = \pm 2$ ，我们设概率 $p(q,r,s,t)$ 为 $Q=q, R=r, S=s, T=t$ 的概率，这四个变量相互独立，我们求一下它的期望：

$$
\begin{aligned}
E(QS+RS+RT-QT) &= \sum_{qrst} p(q,r,s,t)(qs+rs+rt-qt) \\
&\le \sum_{qrst} p(q,r,s,t) \times 2 \\
&= 2
\end{aligned}
$$

同时：

$$
\begin{aligned}
E(QS+RS+RT-QT) &= \sum_{qrst} p(q,r,s,t)qs + \sum_{qrst} p(q,r,s,t)rs \\
&+ \sum_{qrst} p(q,r,s,t)rt - \sum_{qrst} p(q,r,s,t)qt \\
&= E(QS) + E(RS) + E(RT) - E(QT)
\end{aligned}
$$

因此，我们有贝尔不等式 $E(QS) + E(RS) + E(RT) - E(QT) \le 2$ ，这个结果也被称为CHSH不等式（也来源于人名）。

参考资料：yangdaixian，CHSH不等式以及它对量子力学的贡献，
https://blog.csdn.net/yangdaixian/article/details/110287686

4. 世纪之争

- CHSH不等式的优点
  - 测量方向（Q, R）可以在一个子系统上由Alice完成
  - 测量方向（S, T）可以在一个子系统上由Bob完成
  - 上述两个子系统可以位于空间相隔很远的地点

参考资料：张天蓉，世纪幽灵：走进量子纠缠（第二版），中国科学技术大学出版社，2020年

Conclusion

1. Tensor Product
   - Tensor product allows parallel action
2. Assembling System
   - Tensor product of states and acts, graphs and matrices
3. Assembling Quantum System
   - Assembling of independent quantum systems have the tensor product as its state space
4. 世纪之争
   - 玻爱之争
   - EPR佯谬
   - Bell不等式
   - CHSH不等式